

# Non-Markovian quantum trajectories: an exact result

Angelo Bassi<sup>1,\*</sup> and Luca Ferialdi<sup>1,†</sup>

<sup>1</sup>*Dipartimento di Fisica Teorica, Università di Trieste, Strada Costiera 11, 34014 Trieste, Italy.  
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Strada Costiera 11, 34014 Trieste, Italy.*

We analyze the non-Markovian stochastic Schrödinger equation describing a particle subject to spontaneous collapses in space (in the language of collapse models), or subject to a continuous measurement of its position (in the language of continuous quantum measurement). For the first time, we give the explicit general solution for the free particle case ( $H = p^2/2m$ ), and discuss the main properties. We analyze the case of an exponential correlation function for the noise, giving a quantitative description of the dynamics and of its dependence on the correlation time.

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The theory of non-Markovian quantum dynamics is a subject of growing interest, both from the theoretical point of view, as well as for its experimental implications [1]. On the more theoretical side, interest ranges from the theory of open quantum systems [2, 3, 4], to the theory of continuous quantum measurement [5], to quantum trajectories [6, 7, 8], to models of spontaneous wave function collapse [9, 10, 11, 12]. With particular reference to the latter, recent investigations [13] have shown that they might need to be generalized to non-white noises, in order to be compatible with current experimental constraints.

Unlike the theory of Markov dynamics in Hilbert space, which has been deeply investigated and well understood, the theory of non-Markovian quantum dynamics is still under construction. Important results have been already obtained [3, 4]. With particular reference to stochastic Schrödinger equations (SSEs) in Hilbert spaces, these have been formally generalized to non-Markovian noises [7], but explicit results have been obtained only for simple systems [8], or through approximation schemes [12, 14].

In the Markovian case, among all SSEs, the following equation,

$$d\phi_t = \left[ -\frac{i}{\hbar} H dt + \sqrt{\lambda} q dW_t - \frac{\lambda}{2} q^2 dt \right] \phi_t, \quad (1)$$

has received considerable attention [15, 16, 17, 18, 19, 20, 21, 22].  $q$  is the position operator of the particle,  $H$  its quantum Hamiltonian,  $W_t$  a standard Wiener process defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{Q})$  and  $\lambda$  is a positive coupling constant [23]. The evolution described by Eq. (1) is manifestly non-unitary, the non-Schrödinger terms being devised in order to reproduce the collapse of the wave function [24].

The reason why Eq. (1) is so popular is that it represents an excellent compromise between mathematical simplicity and physical adequacy. From the mathematical point of view, it is simple enough to be analyzed in great detail [19, 20, 21]. From the physical point of view instead, it represents a very realistic model describing a quantum particle subject to spontaneous collapses

in space (within collapse models [15, 20]), or a particle whose position is continuously measured by an appropriate device (within the theory of continuous quantum measurement [16, 17]), or a particle coupled to an environment via its position (within the theory of open quantum systems [18]). In all fields of applicability, Eq. (1) has been used to get a deep insight into the dynamics of more complicated physical situations.

It is then of primary interest to study the generalization of Eq. (1) to the non-Markovian case. Such a generalization has been first proposed in [7], and reads:

$$\frac{d}{dt}\phi_t = \left[ -\frac{i}{\hbar} H + \sqrt{\lambda} q w_t - 2\sqrt{\lambda} q \int_0^t ds \alpha(t, s) \frac{\delta}{\delta w_s} \right] \phi_t, \quad (2)$$

where now  $w_t$  is a Gaussian non-white noise defined on  $(\Omega, \mathcal{F}, \mathbb{Q})$ , having zero average and the correlation function  $\alpha(t, s)$ . The non-Markovian character is clearly displayed by the third term, which depends on the whole past history. For this reason, technically speaking, the integration should begin at  $s = -\infty$ . Here we are making the assumption, which actually is an approximation, that the state of the system at time 0 suffices to unfold the subsequent evolution. This would be the case, e.g., if the system has reached an equilibrium configuration which is independent of the way it has been reached, and one is interested in studying what happens if at time 0 the system is driven away from it by a sudden interaction.

Setting Eq. (2) has represented a very important achievement. However it remains still somewhat formal, as no explicit solutions are known. In this Letter we present a recent result, whose technical details are reported in [25]: for the first time, the explicit expression of the Green's function associated to Eq. (2) has been computed, in the case of a free particle ( $H = p^2/2m$ ), and its properties have been analyzed in detail. The technique which has been used can be straightforwardly generalized to include linear and quadratic potentials (thus bounded systems can also be studied). More complicated situations can be analyzed through a perturbation expansion on  $\sqrt{\lambda}$ .

*The Green's function.* In [7] it was first shown that

the Green's function  $G(x, t; x_0, 0)$  associated to Eq. (2) allows for the following path-integral representation:

$$G(x, t; x_0, 0) = \int_{q(0)=x_0}^{q(t)=x} \mathcal{D}[q] e^{\mathcal{S}[q]}, \quad (3)$$

where the 'action'  $\mathcal{S}[q]$ , which is not standard, having both a real and an imaginary part, is:

$$\mathcal{S}[q] = \int_0^t ds \left[ \frac{im}{2\hbar} q_s'^2 + \sqrt{\lambda} q_s w_s - \lambda q_s \int_0^t dr \alpha(s, r) q_r \right]. \quad (4)$$

We have computed the path-integral in (3) using the polygonal approach of Feynman [26, 27]. The calculation is long, in particular due to the last term which contains a double integration reflecting the non-Markovian character of the evolution; nevertheless the computation can be carried out exactly. We report on the final result, focusing on the case of a time-translation invariant noise ( $\alpha(t, s) = \alpha(|t - s|)$ ), which is sufficient for most physical purposes. In this case, the Green's function becomes [25]:

$$G(x, t; x_0, 0) = \sqrt{\frac{m}{2i\pi\hbar t u(t)}} \cdot \exp \left[ -\mathcal{A}_t(x_0^2 + x^2) + \mathcal{B}_t x_0 x + \mathcal{C}_t x_0 + \mathcal{D}_t x + \mathcal{E}_t \right]. \quad (5)$$

The first two coefficients  $\mathcal{A}_t$  and  $\mathcal{B}_t$  are deterministic functions of time and are defined as follows:

$$\mathcal{A}_t = \frac{im}{2\hbar} f_t'(0), \quad \mathcal{B}_t = \frac{im}{\hbar} f_t'(t), \quad (6)$$

while the remaining coefficients  $\mathcal{C}_t$ ,  $\mathcal{D}_t$  and  $\mathcal{E}_t$  depend also on the noise  $w_t$  through the expressions:

$$\mathcal{C}_t = -\frac{im}{2\hbar} h_t'(0) + \frac{\sqrt{\lambda}}{2} \int_0^t dl w_l f_t(l), \quad (7)$$

$$\mathcal{D}_t = \frac{im}{2\hbar} h_t'(t) + \frac{\sqrt{\lambda}}{2} \int_0^t dl w_l f_t(t-l), \quad (8)$$

$$\mathcal{E}_t = \frac{\sqrt{\lambda}}{2} \int_0^t dl w_l h_t(l). \quad (9)$$

(Here above and in the following, the symbol ' denotes differentiation with respect to the variable within parenthesis.) The random function  $h_t(s)$  satisfies the following non-homogeneous integro-differential equation:

$$\frac{im}{2\hbar} h_t''(s) + \lambda \int_0^t dr \alpha(s, r) h_t(r) = \frac{\sqrt{\lambda}}{2} w_s, \quad (10)$$

with boundary conditions  $h_t(0) = h_t(t) = 0$ . The function  $f_t(s)$  instead satisfies the homogeneous equation associated to Eq. (10), with boundary conditions  $f_t(0) = 1$ ,  $f_t(t) = 0$ . Also the function  $u(t)$  can be given an analytic expression in terms of the solution of an integro-differential equation; since however the whole square root in (5) represents a global factor whose real part loses importance when normalizing the wave function, and whose

imaginary part represents an uninteresting global phase factor, we omit to write the explicit expression of  $u(t)$ .

Eqs. (5)-(10) represent our main result, from which the subsequent discussion follows. One should notice the quite remarkable fact that we have been able to compute the Green's function associated to Eq. (2) (which can be applied to any  $\mathcal{L}^2$  initial state, giving its time evolution), while in general non-Markovian dynamics do not allow for such a thing as the Green's function. This fact is less surprising if one looks back at how Eq. (2) was derived [6]: first the evolution was set by means of a propagator, and only afterwards the associated differential equation was deduced.

Another relevant observation to make is that the structure of the non-Markovian Green's function  $G(x, t; x_0, 0)$  is the same as the corresponding Markovian one [21, 28], and of course reduces to it in the white-noise limit, as proven in [25]. In particular, the exponent is quadratic in the variables  $x_0$ ,  $x$ , and the coefficients associated with the quadratic terms do not depend on the noise. This fact has two important consequences: first, the shape of Gaussian states is preserved during the evolution; second, their spread evolves deterministically in time. We will come back on these points later. Since more general states can be written as superpositions of Gaussian states, these facts suggest that any reasonable initial state converges almost surely to a Gaussian state with a fixed spread both in position as well as in momentum. This property holds in the Markovian case, and has been subject of an intense investigation [17, 18, 20, 21, 28, 29]. It would be important to check it also in a non-Markovian setting.

As a second relevant consequence of Eq. (5), one can verify that the following ansatz:

$$\frac{\delta}{\delta w_s} \phi_t = [q a_t(s) + p b_t(s) + c_t(s)] \phi_t, \quad (11)$$

first proposed in [8], is correct; the three coefficients have the following time dependence [25]:

$$a_t(s) = f_t(t-s) + \frac{f_t'(0)}{f_t'(t)} f_t(s), \quad b_t(s) = \frac{1}{m} \frac{f_t(s)}{f_t'(t)}, \quad (12)$$

$$c_t(s) = h_t(s) - \frac{f_t(s)}{2f_t'(t)} \left( h_t'(t) + \frac{i\sqrt{\lambda}\hbar}{m} \int_0^t dl w_l f_t(t-l) \right). \quad (13)$$

One can then replace the functional derivative appearing in (2) with (11), giving the non-Markovian equation a less cumbersome expression. The form (11) for the functional derivative should make it clear that the non-Markovian term of Eq. (2) depends on the interplay between the Hamiltonian and the collapse terms, since a term proportional to  $p$  appears, which can come only from the free part of the evolution. This is the ultimate reason why the functional derivative can be computed explicitly

only when all operators appearing in Eq. (2) commute with each other [10], or in simple enough cases like ours.

One can further prove [25] that the mean position  $\mathbb{E}_{\mathbb{Q}}[\langle q \rangle_t]$  and mean momentum  $\mathbb{E}_{\mathbb{Q}}[\langle p \rangle_t]$  evolve according to the classical laws. Moreover, the fluctuations of the position of the particle around the average, measured by  $\mathbb{V}_q := \sqrt{\mathbb{E}_{\mathbb{Q}}[\langle q \rangle_t - \mathbb{E}_{\mathbb{Q}}[\langle q \rangle_t]]^2}$ , scale with the inverse square root of its mass; this means that, the bigger the system, the less random the motion within a given time interval.

*Exponential correlation function.* The explicit form of the coefficients  $\mathcal{A}_t$ – $\mathcal{E}_t$  defining the Green's function depend on the solution  $h_t(s)$  of Eq. (10) and on the solution  $f_t(s)$  of the corresponding homogeneous equation. In general, this equation cannot be solved explicitly, though a perturbation expansion is always possible, which gives meaningful results to first orders in  $\lambda$ . Nevertheless, the solution can be found for particular types of correlation functions [30]. Among these, the physically most meaningful example is the exponential correlation function:

$$\alpha(t, s) = (\gamma/2)e^{-\gamma|t-s|}, \quad (14)$$

where  $\gamma$  is the inverse of the correlation time.

With this choice for  $\alpha(t, s)$ , the homogeneous equation for  $f_t(s)$  can be solved as follows. By differentiating twice Eq. (10) with  $w_s = 0$ , one can transform the integro-differential equation into the fourth-order differential equation [25]:

$$f_t''''(s) - \gamma^2 f_t''(s) + i\gamma^2 \omega^2 f_t(s) = 0, \quad (15)$$

where  $\omega = 2\sqrt{\hbar\lambda/m}$ . The general solution is  $f_t(s) = \sum_{k=1}^2 [f_{t,k} \sinh v_k s + g_{t,k} \cosh v_k s]$ , where  $f_{t,k}$ ,  $g_{t,k}$  are determined by the boundary conditions, and  $v_1$ ,  $v_2$  are the two non-symmetric roots of the bi-quadratic characteristic polynomial associated to Eq. (15):

$$v_{1,2} = \sqrt{(\gamma^2 \pm \zeta)/2}, \quad \zeta = \sqrt{\gamma^4 - 4i\gamma^2 \omega^2}. \quad (16)$$

Two boundary conditions are already given:  $f_t(0) = 1$  and  $f_t'(0) = 1$ . The other two conditions can be recovered [30] from the procedure which led to Eq. (15) and read:  $f_t'''(0) = \gamma f_t''(0)$  and  $f_t'''(t) = -\gamma f_t''(t)$ . Inserting these conditions, one obtains:

$$f_t(s) = \frac{\sum_k [r_t^k \sinh v_k(t-s) + u_t^k \cosh v_k(t-s) - u_s^k]}{\sum_k [2c + r_t^k \sinh v_k t + u_t^k \cosh v_k t]}, \quad (17)$$

with  $k = 1, 2$  and where  $r_t^k = a_k \cosh v_k t + b_k \sinh v_k t$  and  $u_t^k = d_k \sinh v_k t - c \cosh v_k t$ ; we have also defined:  $a_k = \gamma v_k^3 [v_k^2 + (-1)^k \zeta]$ ,  $b_k = v_k^2 [v_k^4 + (-1)^k \gamma^2 \zeta]$ ,  $c = v_1^3 v_2^3$ ,  $d_k = -\gamma v_k^3 v_{\bar{k}}^2$ , with  $\bar{k} = 2$  if  $k = 1$ ,  $\bar{k} = 1$  if  $k = 2$ .

The function  $h_t(s)$  can be found in a similar way, though its expression is more complicated, as  $h_t(s)$  solves the whole inhomogeneous equation. Taking into account the boundary conditions,  $h_t(s)$  takes the form:

$h_t(s) = h_t^p(s) - h_t^p(t)f_t(t-s)$ , where  $h_t^p(s)$  is a particular solution of (10), namely:

$$h_t^p(s) = -\frac{i\sqrt{\lambda}\hbar}{m} \int_0^s \bar{f}_s(l) (w_l'' - \gamma^2 w_l) dl, \\ \bar{f}_s(l) = \frac{\sinh v_1(s-l)}{v_1} - \frac{\sinh v_2(s-l)}{v_2}. \quad (18)$$

The problem has been completely solved. One can check that in the white-noise limit  $\gamma \rightarrow \infty$  ( $\alpha(t, s) \rightarrow \delta(t-s)$ ), one recovers the well-known Markovian expressions.

*Evolution of Gaussian states.* The analysis of Gaussian states is particularly useful in order to understand the behavior of a typical physical state. As previously anticipated, the shape of Gaussian wave functions does not change in time. In fact, an initial state:

$$\phi_0(x) = \exp[-\alpha_0 x^2 + \beta_0 x + \gamma_0], \quad (19)$$

preserves its functional dependence on  $x$ , while the complex parameters  $\alpha_0$ ,  $\beta_0$  and  $\gamma_0$  evolve in time as follows:

$$\alpha_t = \mathcal{A}_t - \frac{\mathcal{B}_t^2}{4(\alpha_0 + \mathcal{A}_t)}, \quad \beta_t = -\frac{\mathcal{C}_t + \beta_0}{4(\alpha_0 + \mathcal{A}_t)} + \mathcal{D}_t \\ \gamma_t = \gamma_0 + \mathcal{E}_t + \frac{(\mathcal{C}_t + \beta_0)^2}{4(\alpha_0 + \mathcal{A}_t)}. \quad (20)$$

Analyzing the above expressions with the help of Eqs. (6)–(9), one immediately sees that the evolution of  $\alpha_t$  is deterministic, while  $\beta_t$  and  $\gamma_t$  have stochastic terms. This means that, like in the white-noise case, both the spread in position and in momentum of  $\phi_t(x)$ , which are given by  $\alpha_t$ , evolve deterministically in time. On the other hand, both the mean position and the mean momentum, which depend both on  $\alpha_t$  and  $\beta_t$ , have stochastic components; their stochastic averages instead evolve according to classical laws, as we have already anticipated.

We focus now our attention on the spread in position  $\sigma(t) = 1/2\sqrt{\alpha_t^R}$ , in the case of the exponential correlation function treated before. Fig. 1 shows how the spread evolves, for different values of  $\gamma$ . Qualitatively the behavior is the same for any  $\gamma$ : the wave function shrinks in space, reaching an asymptotic finite value. On a more quantitative level, we see that the stronger  $\gamma$ , the faster the collapse. One can also notice that the collapse is effective starting with relatively small values of  $\gamma$ : a value  $\gamma \sim 10 \text{ sec}^{-1}$  already ensures that after about  $10^{-3} \text{ sec}$  the wave function has collapsed below  $10^{-5} \text{ cm}$ , which is the threshold chosen by GRW [24], below which a state can be considered as localized. This means that the possibility opens for non-Markovian models to be as effective as the corresponding white-noise models as far as the collapse process is concerned, but, at the same time, to give different physical predictions regarding specific experimental situations. This possibility has first been suggested in [13].

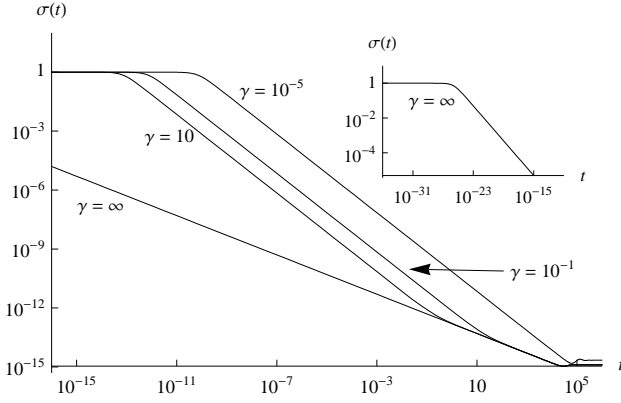


FIG. 1: Time evolution, under the assumption of an exponential correlation function, of the spread in position  $\sigma(t)$  of a Gaussian wave function.  $\sigma(0)$  has been set = 1 m. The value  $\gamma = \infty$  corresponds to the Markovian case. The other parameters have been chosen as follows:  $m = 1$  Kg,  $\lambda_0 = 10^{-2}$  m<sup>-2</sup> sec<sup>-1</sup>. Time is measured in sec, distances in m.

From the previous expressions one can explicitly compute the asymptotic value of  $\alpha_t$ , which is:

$$\alpha_\infty = \lim_{t \rightarrow \infty} \alpha_t = -\frac{im}{2\hbar}(v_1 + v_2 - \gamma). \quad (21)$$

The quantity  $1/2\sqrt{\alpha_\infty^R}$  is the final spread in position to which all Gaussian states (and, reasonably, any initial state) converge to, in the long-time limit.

*Conclusion.* We have computed for the first time the Green's function associated to the motion of a free particle as described by Eq. (2), from which the entire non-Markovian dynamics can be unfolded. We have analyzed the physically important case of an exponential correlation function. By studying Gaussian states, we have seen how the collapse occurs, and have derived an exact expression for the asymptotic spread. The tools we have employed to derive the above results are flexible and can be applied to more complex physical situations.

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\* Electronic address: bassi@ts.infn.it

† Electronic address: ferialdi@ts.infn.it

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